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LETTER TO THE EDITOR

The leading asymptotic term for the scattering diagram in the problem of diffraction by a narrow circular impedance cone

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Abstract. The leading term for the scattering diagram, in the scalar problem concerning the diffraction of a plane wave by a narrow circular impedance cone, is obtained. Although, up to now, the problem cannot be solved in an explicit form, its reduction to a non-oscillating integral equation has been recently developed. It is used here in order to determine formal asymptotics of the scattering diagram by means of a perturbation method.

It is well known that one can obtain closed form expressions of the scattering diagram for a narrow circular cone (Felsen 1955, 1957) and, more generally, for an arbitrary shaped narrow cone (Babich 1996) if Dirichlet or Neumann boundary conditions are imposed on its surface, when the wave field is governed by the scalar wave (Helmholtz) equation. In comparison, as mentioned by Jones (1964), the problem is much more difficult if we consider mixed boundary conditions of constant impedance type. This type of boundary condition was extensively used since the works of Leontovich, and it has been the subject of many analytical developments (see for instance, Maliuzhinets (1958), and also some recent results, Bernard (1998), Lyalinov (1997), for the diffraction by 2D wedge-shaped singularity). It is worth noting that separation of variables fails to give a discrete basis of functions satisfying a constant impedance condition for obstacles with edge or vertex, even in the case of an impedance cone with a circular cross section. However, an analytical method was recently developed (Bernard 1997) for this case, reducing the problem to the unique solution of an integral equation with non-oscillating kernel. In this letter we exploit these results and develop the leading asymptotic term of the scattering diagram for a narrow impedance cone. As is known, the scattering diagram (or diffraction coefficient) is one of the most important characteristics in the theory of diffraction, in particular because of its use in high frequency techniques (Keller 1962). A harmonic time dependence $e^{i\omega t}$ is from now on assumed and suppressed throughout.

We consider that an incoming plane wave U^i illuminates a circular cone, with

$$U^i(R,\theta,\varphi) = e^{ikR(\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos\varphi)} \tag{1}$$

where R, θ, φ are the spherical coordinates (see figure 1) and k is a wavenumber with $|\arg(ik)| < \pi/2$. We assume that the total radiated field $u = U^i + U$ satisfies, in the domain

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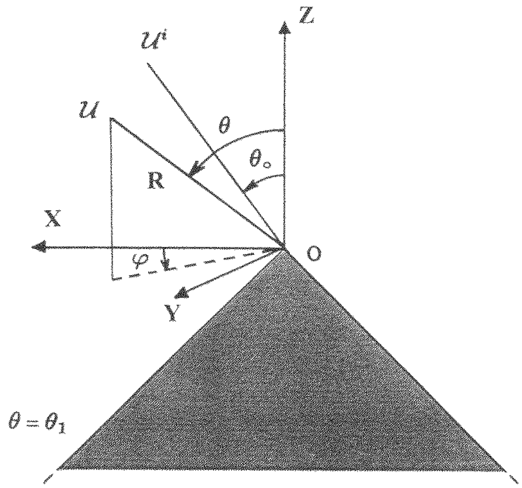

Figure 1. Geometry of the cone.

$\theta < \theta_1$, the Helmholtz equation $(\Delta + k^2)u = 0$, and, on the cone's surface $\theta = \theta_1 > \pi/2$, the constant impedance boundary condition $(\frac{\partial}{R\partial\theta} + ik\sin\zeta)u = 0$, where the relative impedance $\sin\zeta$ is a constant with $\mathrm{Re}(\sin\zeta) > 0$. Using the results of Bernard (1997), the scattered field U in the problem of diffraction by an impedance circular cone can be determined in the form of Fourier series

$$U(R,\theta,\varphi) = \sum_{n=-\infty}^{\infty} i^n \mathcal{E}_n(R,\theta)\mathrm{e}^{-\mathrm{i}n\varphi} \qquad i^n\mathcal{E}_n = i^{-n}\mathcal{E}_{-n} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} U\mathrm{e}^{\mathrm{i}n\varphi}\,\mathrm{d}\varphi \tag{2}$$

with, for $n = \mu - \frac{1}{2} \geqslant 0$,

$$\mathcal{E}_{\mu-1/2}(R,\theta) = \int_0^{i\infty} G(t)(\mp i)^{\mu-1/2} P_{t-1/2}^{-\mu+1/2}(\cos[\theta \mp i0^+])K_t(ikR)(ikR)^{-1/2}\,\mathrm{d}t/i \tag{3}$$

where $P_{t-1/2}^{-\mu+1/2}(z)$ is the associated Legendre function and $K_\nu(z)$ is Macdonald's (modified Bessel) function. The representation (3) has been shown to exist when the angles θ and θ_0, respectively angle of observation and angle of incidence of the incoming plane wave, satisfy the inequalities

$$\theta_1 - \theta_0 > \pi/2 \qquad \theta_1 - \theta + \theta_1 - \theta_0 - \pi/2 > |\arg(ik)| \tag{4}$$

considering that $U = \mathrm{O}(R^l)$ with $l > -\frac{1}{2}$ in vicinity of the apex, and $U = \mathrm{O}(\mathrm{e}^{\mathrm{i}kR\cos(\epsilon+\pi/2)})$ as R tends to infinity, $\epsilon(\theta,\theta_0) = \min(\pi/2, 2\theta_1 - (\theta + \theta_0 + \pi/2))$.

Our aim is to determine in closed-form the leading term of the scattering diagram F of the wave scattered by the vertex of a narrow impedance cone. This function, related to the expression of U as R tends to infinity, is defined by

$$F(\theta,\varphi) = \sum_{n=-\infty}^{\infty} i^{2n} f(n+\tfrac{1}{2},\theta)\mathrm{e}^{-\mathrm{i}n\varphi} \tag{5}$$

where f follows from the expression

$$\mathcal{E}_{\mu-1/2} = f(\mu,\theta)\frac{\mathrm{e}^{-\mathrm{i}kR}}{ikR}\mathrm{e}^{\mathrm{i}\pi/2(\mu-1/2)}(1 + \mathrm{O}(1/kR))$$

as we use the leading asymptotic term of the modified Bessel function coming from

$$K_\nu(z) = \sqrt{\frac{\pi}{2}}\frac{\exp(-z)}{\sqrt{z}}(1 + \mathrm{O}(z^{-1})) \qquad |z| \gg 1 \quad |\arg z| < \pi$$

in the expression (3), the resulting integral then converging as (Bernard 1997)

$$\theta_1 - \theta + \theta_1 - \theta_0 - \pi/2 > \pi/2. \tag{6}$$

This condition properly describes a domain where the incident and the tip diffracted waves, but not the geometrical optics reflected waves, exist in the far field. We note that this domain corresponds to take k real in the inequalities (4).

The function $G(t)$ in (3) is determined through the unique solution, in the space $L^1(iR)$, of the integral equation (Bernard 1997†)

$$\mathcal{R}(\nu) = \frac{\sin\zeta}{2}\int_{-i\infty}^{i\infty} W(t)\mathcal{R}(t)\frac{\sin\pi t}{\cos\pi t + \cos\pi\nu}\,\mathrm{d}t + \mathcal{S}^i(\nu) \tag{7}$$

with

$$W(t) = -it\,\left.\frac{\mathcal{P}(\mu,t,\theta)}{\partial\mathcal{P}/\partial\theta}\right|_{\theta=\theta_1} \tag{8}$$

$$\mathcal{P}(\mu,t,\theta) = (\mp i)^{\mu-1/2}P_{t-1/2}^{-\mu+1/2}(\cos[\theta \mp i0^+])$$

$$\mathcal{R}(t) = \frac{\pi}{4t\sin\pi t}G(t)\,\left.\frac{\partial\mathcal{P}}{\partial\theta}\right|_{\theta=\theta_1}. \tag{9}$$

The free term $\mathcal{S}^i$ of the integral equation (7) is attached to the incident field U^i. It can be expanded as follows (Bernard 1997)

$$\begin{aligned}\mathcal{S}^i(\nu) &= -\frac{1}{2}\int_{-i\infty}^{i\infty}\left[\sin\zeta\mathcal{R}^i(t)\,\left.\frac{it\mathcal{P}^i}{\partial\mathcal{P}^i/\partial\theta}\right|_{\theta=\theta_1} + \frac{i}{2}(\mathcal{R}^i(t+1) - \mathcal{R}^i(t-1))\right]\frac{\sin\pi t}{\cos\pi t + \cos\pi\nu}\,\mathrm{d}t \\ &= -\frac{1}{2}\int_{-i\infty}^{i\infty}\sin\zeta\mathcal{R}^i(t)\,\left.\frac{it\mathcal{P}^i}{\partial\mathcal{P}^i/\partial\theta}\right|_{\theta=\theta_1}\frac{\sin\pi t}{\cos\pi t + \cos\pi\nu}\,\mathrm{d}t - \mathcal{R}^i(\nu)\end{aligned} \tag{10}$$

with

$$\begin{aligned}\mathcal{P}^i(\mu,t,\theta) &= (\mp i)^{\mu-1/2}P_{t-1/2}^{-\mu+1/2}(\cos[\pi - (\theta \pm i0^+)]) \\ \mathcal{R}^i(t) &= \frac{\pi}{4t\sin\pi t}G^i(t)\,\left.\frac{\partial\mathcal{P}^i}{\partial\theta}\right|_{\theta=\theta_1} \\ G^i(t) &= -\frac{\sqrt{2}}{\sqrt{\pi}^3}t\sin(\pi t)\Gamma(\mu+t)\Gamma(\mu-t)P_{t-1/2}^{-\mu+1/2}(\cos[\theta_0 + i0^+]).\end{aligned} \tag{11}$$

The principal formulae concerning the reduction of the scalar problem *being stated*, we can search an explicit calculus for $\mathcal{R}$ and then for the scattering diagram F.

It is possible to show that, when θ_1 tends to π (narrow cone), the integral equation (7) can be solved with a perturbation method. In order to obtain the leading term of a solution of this equation, we use asymptotics (Erdelyi *et al* 1953) of the functions $\mathcal{P}$ and $\mathcal{P}^i$. As $\theta_1 \to \pi$, we have

$$\begin{aligned}\mathcal{P}(\tfrac{1}{2},\nu,\theta_1) &\sim \frac{\sin\pi(\nu-\frac{1}{2})}{\pi}\left(\ln\left(\frac{1+\cos(\theta_1 \mp i0^+)}{2}\right) + \gamma + 2\psi(\nu-\tfrac{1}{2}) + \pi\cot\pi(\nu-\tfrac{1}{2})\right) \\ &= \frac{\sin\pi(\nu-\frac{1}{2})}{\pi}(2\ln(\pi-\theta_1) + \mathrm{O}_\nu(1))\end{aligned} \tag{12}$$

† In this work replace χ_d by R_d in the expression of I_2 p 39, in (3.31) p 41 and in section (b) of appendix 12; change $\sin\zeta$ into $-\sin\zeta$ p 49 (after the first colon) to p 53; replace p 53, in first sentence W_n by $-W_{e,\delta}$, and in the coupled integral equations $-\frac{1}{2}$ by $\frac{1}{2}$.

where γ is the Euler constant and $\psi(\nu)$ is the digamma function, and

$$\mathcal{P}(\mu,\nu,\theta_1) \sim \frac{\Gamma(\mu-\frac{1}{2})(1+\cos(\theta_1))^{-\mu/2+1/4}}{2^{-\mu/2+1/4}\Gamma(\mu+\nu)\Gamma(\mu-\nu)}$$
$$= C_{\mu,\nu}(\pi-\theta_1)^{-\mu+1/2}(1+\mathrm{O}_{\mu,\nu}(\pi-\theta_1)^2) \qquad \mu > \tfrac{1}{2}. \tag{13}$$

In the same manner, we have

$$\mathcal{P}^i(\tfrac{1}{2},\nu,\theta_1) = 1-(\nu^2-\tfrac{1}{4})(\pi-\theta_1)^2/4 + \mathrm{O}_\nu((\pi-\theta_1)^4) \tag{14}$$

and

$$\mathcal{P}^i(\mu,\nu,\theta_1) \sim \frac{(1-\cos(\pi-\theta_1))^{\mu/2-1/4}}{2^{\mu/2-1/4}\Gamma(\mu+1/2)} = D_{\mu,\nu}(\pi-\theta_1)^{\mu-1/2}(1+\mathrm{O}_{\mu,\nu}(\pi-\theta_1)^2) \tag{15}$$

for $\mu-\frac{1}{2}$ a positive integer. The estimates (12), (13) enable us to conclude that the function $W(t)$ in the kernel of the integral equation has the asymptotics, as $\theta_1 \to \pi$,

$$W(t) \sim it(\pi-\theta_1)\ln(\pi-\theta_1) \tag{16}$$

if $\mu = \frac{1}{2}$, and

$$W(t) \sim it(\pi-\theta_1)/(\tfrac{1}{2}-\mu) \tag{17}$$

for $\mu > \frac{1}{2}$. Analogously we find from (14), (15) that, as $\theta_1 \to \pi$,

$$it\frac{\mathcal{P}^i(1/2,t,\theta_1)}{\partial\mathcal{P}^i/\partial\theta|_{\theta_1}} \sim \frac{2it}{t^2-\frac{1}{4}}(\pi-\theta_1)^{-1} \tag{18}$$

and

$$it\frac{\mathcal{P}^i(\mu,t,\theta_1)}{\partial\mathcal{P}^i/\partial\theta|_{\theta_1}} \sim \frac{it}{\frac{1}{2}-\mu}(\pi-\theta_1) \tag{19}$$

for $\mu-\frac{1}{2}$ a strictly positive integer.

It is now obvious that the integral equation has asymptotically small kernel for a narrow cone, and thus, that previous expressions allow to find the leading term of the solution by a perturbation method. As follows from expression (10) of $\mathcal{S}^i$ in the leading approximation, the solution is then given, except for $|\sin\zeta| = 0$ or ∞, by

$$\mathcal{R}^0(\nu) = -\frac{i\pi\sin\zeta}{8}\int_{-i\infty}^{i\infty}\frac{G^i(t)}{\cos\pi t+\cos\pi\nu}\,\mathrm{d}t \tag{20}$$

where we take $\mu = \frac{1}{2}$ for the expression of G^i. The other terms corresponding to $\mu \neq \frac{1}{2}$ give contributions of lower orders of size with respect to the small parameter. By means of formula (9) for $\mathcal{R}$ and the expression of G^i, we then find, from (20), that the leading term for G is given by

$$G^0(\nu) = i\sqrt{\pi/2}(\pi-\theta_1)\sin(\zeta)\nu\tan(\pi\nu)\int_0^{i\infty}P^0_{t-1/2}(\cos[\theta_0+i0^+])\frac{t\tan(\pi t)}{\cos\pi t+\cos\pi\nu}\,\mathrm{d}t. \tag{21}$$

We then use formula (21) in order to obtain the leading term of the scattering diagram for a narrow impedance cone. We substitute G^0 into expression (3), exploit the asymptotics of the modified Bessel function, and then obtain the leading term F^0 for F. Finally, we come to the expression for the scattering diagram in the leading approximation

$$F^0(\theta,\varphi) = \frac{\pi(\pi-\theta_1)\sin\zeta}{2}\int_0^{i\infty}\int_0^{i\infty}P^0_{t-1/2}(\cos[\theta_0+i0^+])P^0_{\nu-1/2}(\cos[\theta+i0^+])$$
$$\times\frac{t\nu\tan(\pi t)\tan(\pi\nu)}{\cos\pi t+\cos\pi\nu}\,\mathrm{d}t\,\mathrm{d}\nu. \tag{22}$$

Let us recall that we have obtained this result assuming $|\sin\zeta|$ is not equal to 0 or ∞, which signifies more precisely, from (16) and (18), that the approximation by the leading term (22) is valid provided the condition

$$(\pi-\theta_1) \ll |\sin\zeta| \ll 1/|(\pi-\theta_1)\ln(\pi-\theta_1)| \tag{23}$$

is satisfied. We notice that the integral converges when the conditions (6) with $\theta_1 = \pi$ are satisfied and that the diagram does not depend on the coordinate φ in the narrow cone approximation. The expression (22) is also symmetric with respect to θ_0 and θ, which was expected due to the reciprocity theorem. Moreover, it is quite remarkable that it is now possible to calculate a closed form expression of the double integral (22).

For this purpose, we use Fock's representation for the Legendre functions (Fock 1943)

$$P_{ix-1/2}(\cosh v) = (2/\pi)\coth(\pi x)\int_0^\infty \frac{\sin(x[w+v])}{\sqrt{2(\cosh(w+v)-\cosh v)}}\,\mathrm{d}w. \tag{24}$$

After changing the orders of integration, we then come to the expression

$$F^0(\theta,\varphi) = -\frac{2A_0}{\pi^2}\int_0^\infty\int_0^\infty \mathrm{d}w_1\,\mathrm{d}w_2\left[\int_0^\infty\int_0^\infty \times\frac{x\sin(x[w_1+v])\,y\sin(y[w_2+v_0])\,\mathrm{d}x\,\mathrm{d}y}{\sqrt{\cosh(w_1+v)-\cosh v}\sqrt{\cosh(w_2+v_0)-\cosh v_0}(\cosh\pi x+\cosh\pi y)}\right] \tag{25}$$

where $A_0 = (\pi/2)(\pi-\theta_1)\sin\zeta$, $v_0 = -i\theta_0 + 0^+$ and $v = -i\theta + 0^+$. The integrations with respect to w_1, w_2 are due to the respective integral representations (24) of the two Legendre functions. Now, taking into account that $\partial_c(\cos(a[b_1+c]) = -a\sin(a[b_1+c])$, we obtain the expression

$$F^0(\theta,\varphi) = -\frac{2A_0}{\pi^2}\int_0^\infty\int_0^\infty \mathrm{d}w_1\,\mathrm{d}w_2\frac{1}{\sqrt{\cosh(w_1+v)-\cosh v}\sqrt{\cosh(w_2+v_0)-\cosh v_0}} \times\frac{\partial^2}{\partial v\partial v_0}\left[\int_0^\infty\int_0^\infty \frac{\cos(x[w_1+v])\cos(y[w_2+v_0])}{\cosh\pi x+\cosh\pi y}\,\mathrm{d}x\,\mathrm{d}y\right]. \tag{26}$$

For the internal integral on x, we use formula 3.983.2 from (Gradshteyn and Ryzhik 1980), then we compute the integral with respect to y by means of the formula 3.981.5. After differentiation on v and v_0 we can write

$$F^0(\theta,\varphi) = -\frac{2A_0}{\pi^2}\int_0^\infty\int_0^\infty \mathrm{d}w_1\,\mathrm{d}w_2[\sinh(w_2+v_0)\sinh(w_1+v)]\left[\sqrt{\cosh(w_1+v)-\cosh v}\sqrt{\cosh(w_2+v_0)-\cosh v_0}(\cosh(w_1+v)+\cosh(w_2+v_0))^3\right]^{-1}. \tag{27}$$

We introduce the new variables of integration in (27)

$$p = \sqrt{\cosh(w_1+v)-\cosh v} \qquad q = \sqrt{\cosh(w_2+v_0)-\cosh v_0}$$

and come to the double integral

$$F^0(\theta,\varphi) = -\frac{8A_0}{\pi^2}\int_0^\infty\int_0^\infty\left[\frac{1}{(\cos\theta+\cos\theta_0+p^2+q^2)^3}\right]\mathrm{d}p\,\mathrm{d}q. \tag{28}$$

The double integral (28) is easily calculated by means of introducing the polar coordinates. As a result, we then obtain:

$$F^0(\theta,\varphi) = -\frac{(\pi-\theta_1)\sin\zeta}{2(\cos\theta+\cos\theta_0)^2} \tag{29}$$

leading asymptotic term for F as θ_1 tends to π (except for $\sin\zeta = 0$ or ∞), and narrow cone approximation of the scattering diagram as the condition (23) on $\sin\zeta$ is satisfied.

Whereas the analogous results for cones with Dirichlet or Neumann conditions are well known, the formula (29) for a narrow circular impedance cone seems to be new and not considered in the literature. Its simpleness should permit an easy use for further mathematical and physical developments, and for tests of other results such as numerical ones.

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